Components and Exit Times of Brownian Motion in Multiple p-Adic Dimensions

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Joint work with my advisor, David Weisbart: "Components and Exit Times of Brownian Motion in Two or More p-Adic Dimensions."
http://arxiv.org/abs/2210.16429.

Brownian Motion

- The motion of particles suspended in a fluid at dynamic equilibrium
- Statistical Mechanics models (e.g. Einstein, Smoluchowski) as well as Stochastic Process models
- Applications in mathematics, physics, biology, finance...

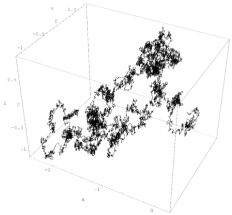

Figure: A realisation of the $\mathbb{R}^3$-valued Wiener Process model [1]

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p-Adic Spaces

- Invented for (analytic) number theory
- Prototypical non-archimedean space
- Admits much richer hierarchy of division rings of finite rank
- Beyond number theory: Applications to physics, neural networks, and more being actively pursued (e.g. Zuniga-Galindo, Khrennikov, Volovich)

The Classical Wiener Process

$$P(W_t \in \Omega \subset \mathbb{R}) = \int_\Omega \frac{1}{\sqrt{2\pi\sigma t}} e^{-\frac{x^2}{2\sigma t}} dx \tag{1}$$

$$u_t = \sigma \Delta u \tag{2}$$

$$P(\vec{W}_t \in \Omega \subset \mathbb{R}^d) = \int_\Omega \frac{1}{(2\pi\sigma t)^{d/2}} e^{-\frac{\|\vec{x}\|^2}{2\sigma t}} d\vec{x} \tag{3}$$

$$\vec{W}_t = \left(W_t^{(1)}, \cdots, W_t^{(d)}\right), \quad W_t^{(i)} \overset{iid}{\sim} W_t \tag{4}$$

A Wiener Process Analogue

- (Varadarajan 1997) Stochastic process from fundamental solution to a pseudo-differential analogue of the heat equation for "a vector space of finite dimension over a non-archimedean locally compact division ring"

$$u_t = \sigma\left(\mathcal{F}^{-1}\mathcal{M}_b\mathcal{F}\right)u, \quad \left(\mathcal{M}_b f\right)(\vec{x}) = \|\vec{x}\|^b f(\vec{x}) \tag{5}$$

- In the concrete case $\vec{X}_t$ valued in $\mathbb{Q}_p^d$, density given by the fundamental solution

$$\rho_d(t,\vec{x}) = \left(\mathcal{F}^{-1}e^{-\sigma t\|\cdot\|^b}\right)(\vec{x}) \tag{6}$$

$$= \int_{\mathbb{Q}_p^d}\chi(\vec{x}\cdot\vec{y})e^{-\sigma t\|\vec{y}\|^b}d\vec{y} \tag{7}$$

$$\|\vec{y}\| = \max\left(|y_1|,\ldots,|y_d|\right)$$

$$|\cdot| \in \{0 < \cdots < p^{-1} < p^0 < p^1 < \cdots\}$$

Failure of Component Independence

Notation:

$$\vec{X}_t = \left(X_t^{(1)}, \ldots, X_t^{(d)}\right), \quad \vec{X}_{t,1} = \left(X_t^{(2)}, \ldots, X_t^{(d)}\right)$$

$$B(r,a) = \{x \in \mathbb{Q}_p : |x-a| \le p^r\}, \quad S(R) = \{x \in \mathbb{Q}_p : |x| = p^R\}$$

For small $t > 0$, integers $r < R$, and $a \in B(R)$:

$$P\left(X_t^{(1)} \in B(r,a) \middle| \vec{X}_{t,1} \in \prod_{k=2}^{d} S(R)\right) \le p^{r-R} < \frac{1}{p} \tag{8}$$

$$P\left(X_t^{(1)} \in B(r,0)\right) > \frac{1}{p} \tag{9}$$

Characterizing the Components

Theorem *(R.R. & D. Weisbart)*

- For any positive real number t, the components of $\vec{X}_t$ are not independent.
- The component processes $X_t^{(i)}$ are identically distributed with the same diffusion parameters as $\vec{X}_t$.

A Key Oscillatory Integral

$$\rho^{(1)}(t,x) = \int_{\mathbb{Q}_p^{d-1}} \rho_d(t,(x,\vec{x}_1))d\vec{x}_1 \tag{10}$$

$$= \int_{\mathbb{Q}_p^{d-1}} \int_{\mathbb{Q}_p^{d}} \chi((x,\vec{x}_1)\cdot\vec{y})e^{-\sigma t\|\vec{y}\|^b}d\vec{y}d\vec{x}_1 \tag{11}$$

$$= \int_{\mathbb{Q}_p} \chi(x\cdot y)e^{-\sigma t|y|^b}dy = \rho_1(t,x) \tag{12}$$

Exit Time Analysis

- Let $Y_t^{(i)}$ be iid with one-dimensional density ρ_1, and define

$$\vec{Y}_t = \left(Y_t^{(1)}, \ldots, Y_t^{(d)}\right)$$

- Extending (Weisbart 2021), set

$$\alpha_d = 1 - \frac{p^b - 1}{p^{b+d} - 1},$$

and for any $\mathbb{Q}_p^d$-valued stochastic process $\vec{Z}$ define

$$\left\{\vec{Z}\right\}_T = \sup_{0 \le t \le T} \|\vec{Z}_t\|$$

- First exit probabilities are the complement of the survival probabilities:

$$P\left(\left\{\vec{Y}\right\}_T \le p^R\right) = e^{-d\sigma\alpha_1 T p^{-Rb}} \xrightarrow{d\to\infty} 0 \tag{13}$$

$$P\left(\left\{\vec{X}\right\}_T \le p^R\right) = e^{-\sigma\alpha_d T p^{-Rb}} \xrightarrow{d\to\infty} e^{-\sigma T p^{-Rb}} \tag{14}$$

Some Future Directions

- Generalizing to the setting of (Varadarajan 1997)
- Fine analysis of the nature of component dependency
- Distinguishing $\vec{X}_t$ and $\vec{Y}_t$ in fixed dimension
- Further connections with p-adic physics and non-archimedean modeling
- Efficient simulation

Bibliography

- "Brownian Motion." In Wikipedia, December 14, 2022. https://en.wikipedia.org/w/index.php?title=Brownian_motion&oldid=1127433565.
- Rajkumar, Rahul, and David Weisbart. "Components and Exit Times of Brownian Motion in Two or More *p*-Adic Dimensions." arXiv, October 28, 2022. http://arxiv.org/abs/2210.16429.
- Varadarajan, V. S. "Path Integrals for a Class of P-Adic Schrödinger Equations." Letters in Mathematical Physics 39, no. 2 (January 1, 1997): 97–106. https://doi.org/10.1023/A:1007364631796.
- Weisbart, David. "Estimates of Certain Exit Probabilities for *p*-Adic Brownian Bridges." arXiv, May 22, 2020. http://arxiv.org/abs/2004.02265.